CLIQUE NUMBERS OF GRAPHS AND IRREDUCIBLE EXACT m-COVERS OF $\mathbb{Z}$

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Abstract. For each $m \ge 1$, we construct a graph $G = (V, E)$ with $\omega(G) = m$ such that

$$\max_{1 \le i \le k} \omega(G[V_i]) = m$$

for arbitrary partition $\{V_1, \ldots, V_k\}$ of V, where $\omega(G)$ is the clique number of G and $G[V_i]$ is the induced graph of G with the vertex set V_i. Using this result, we show that for each $m \ge 2$ there exists an exact m-cover of $\mathbb{Z}$ which is not the union of two 1-covers.

1. Introduction

In his proof of the existence of irreducible exact m-covers of $\mathbb{Z}$ (the notions will be introduced soon), Zhang proved the following graph-theoretic result [21, Lemma 2]:

Theorem 1.1. *For every $m \ge 1$, there exists a graph $G = (V, E)$ satisfying the following properties:*

$\omega(G) = m$, where $\omega(G)$ is the clique number of G, i.e., the maximal order of the complete subgraphs of G . And if the vertex set V is arbitrarily split into two non-empty subsets V_1 and V_2, then

$$\omega(G[V_1]) + \omega(G[V_2]) > \omega(G),$$

where $G[V_i]$ denotes the induced subgraph of G with the vertex set V_i.

In this paper, our main purpose is to give an extension of Zhang's result as follows:

Theorem 1.2. *For every $m \ge 1$ and $k \ge 2$, there exists a graph $G = (V, E)$ with $\omega(G) = m$ satisfying the following property:*

If the vertex set V is arbitrarily split into k subsets $V_1, V_2, \ldots, V_k$, then

$$\max_{1 \le i \le k} \omega(G[V_i]) = \omega(G).$$

For an integer a and a positive integer n, let $a(n)$ denote the residue class $\{x \in \mathbb{Z} : x \equiv a \pmod{n}\}$. For a finite system $\mathcal{A} = \{a_t(n_t)\}_{t=1}^{s}$, define the covering function $w_{\mathcal{A}}$ over $\mathbb{Z}$ by

$$w_{\mathcal{A}}(x) := |\{1 \le t \le s : x \in a_t(n_t)\}|.$$

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If $w_{\mathcal{A}}(x) \geq m$ for each $x \in \mathbb{Z}$, we say that a system $\mathcal{A}$ is an *m-cover* of $\mathbb{Z}$. In particular, we call $\mathcal{A}$ an *exact m-cover* provided that $w_{\mathcal{A}}(x) = m$ for all $x \in \mathbb{Z}$. The covers of $\mathbb{Z}$ was firstly introduced by Erdős [4] and has been investigated in many papers (e.g., [8, 10, 22, 12, 1, 15, 16, 19, 2, 6]).

Suppose that $\mathcal{A}_1$ is an m_1-cover and $\mathcal{A}_2$ is an m_2-cover, then clearly $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ forms an (m_1+m_2)-cover. Conversely, Porubský [11] asked whether for each $m \geq 2$ there exists an exact m-cover of $\mathbb{Z}$ which cannot be split into an exact n-cover and an exact $(n-m)$-cover with $1 \leq n < m$. Choi gave such a example for $m = 2$:

$$\mathcal{A} = \{1(2); 0(3); 2(6); 0, 4, 6, 8(10); 1, 2, 4, 7, 10, 13(15); 5, 11, 12, 22, 23, 29(30)\}.$$

In [21], using Theorem 1.1, Zhang gave an affirmative answer to Porubský's problem. This shows that the results on m-covers of $\mathbb{Z}$ is essential. In [20], Sun established a connection between m-covers of $\mathbb{Z}$ and zero-sum problems in abelian p-groups. For more related results, the readers may refer to [14, 18, 17]

On the other hand, for each $m \geq 2$, Pan and Sun [9, Example 1.1] constructed an m-cover of $\mathbb{Z}$ (though not exact) which even is not the union of two 1-covers! As an application of Theorem 1.2, we have a common extension of the above two results:

Theorem 1.3. *For each $m \geq 2$, there exists an exact m-cover of $\mathbb{Z}$ which is not the union of two 1-covers.*

We shall prove Theorem 1.2 in the next section, and the proof of Theorem 1.3 will be given in Section 3.

2. Proof of Theorem 1.2

Lemma 2.1. *Suppose that $G = (V, E)$ is a connected simple graph and v_0 is a vertex of G. Then there exists an oriented graph $\overrightarrow{G}$ arising from G, which satisfies that:*

(i) $\overrightarrow{G}$ doesn't contains any directed cycle.

(ii) For any vertex $u \in V \setminus \{v_0\}$, there exists a directed path of $\overrightarrow{G}$ from v_0 to u.

Proof. We use induction on $|V|$. There is nothing to do when $|V| = 1$ or 2. Now assume that $|V| > 0$ and our assertion holds for any smaller value of $|V|$. Let $V' = V \setminus \{v_0\}$ and $G' = G[V']$. Suppose that $v_1, \ldots, v_s \in V'$ are all vertex adjacent to v_0 in G. By the induction hypothesis, there exists an oriented graph $\overrightarrow{G'}$ obtained from G', satisfying the properties (i) and (ii) for the vertex v_1. Now we direct the edge v_0v_i from v_0 to v_i for $1 \leq i \leq k$, and preserve the direction of each edge in $\overrightarrow{G'}$. Thus we obtain an oriented graph $\overrightarrow{G}$. Clearly $\overrightarrow{G}$ doesn't contain any directed cycle since v_0 can't lie in any directed cycle. And for any $u \in V \setminus \{v_0, v_1\}$, since there exists a directed path of $\overrightarrow{G'}$ from v_1 to u, the property (ii) is also satisfied. □

Lemma 2.2. *For every $k \geq 1$, we can construct a k-chromatic graph without any triangle.*

Proof. The reader may refer to [7] (or [3, Chapter 5, Exercise 23]) for the construction of such graph. In fact, with help of his probabilistic method, Erdős [5]

proved that there exist the graphs having arbitrarily large girths and chromatic numbers. $\square$

Proof of Theorem 1.2. Let $K = (V_K, E_K)$ be a $(k+1)$-chromatic graph without any triangle. Let u_0 be a vertex of K. Then there exists an oriented graph $\overrightarrow{K}$ arising from K, which satisfies the properties (i) and (ii) of Lemma 2.1 for the vertex u_0. Let $n = |V_K|$ and suppose that $u_0, u_1, \ldots, u_{n-1}$ are all vertices of K. For $1 \le i \le n-1$, let l_i denote the length of the longest directed path from u_0 to u_i in $\overrightarrow{K}$. By the property (ii) of Lemma 2.1, these l_i are well-defined. Let $l = \max_{1\le i\le n-1} l_i$, and for $1 \le j \le l$ let

$$D_j = \{1 \le i \le n-1;\ l_i = j\}$$

In particular, we set $D_0 = \{0\}$. For $1 \le i \le n-1$, let

$$A_i = \{0 \le i' \le n-1 : \overrightarrow{u_{i'}u_i} \text{ lies in } \overrightarrow{K}\},$$

where we denote by $\overrightarrow{xy}$ the directed edge from x to y. In particular, we set $A_0 = \emptyset$.

Lemma 2.3. *For $1 \le j \le l$, we have*

$$\bigcup_{u_i \in D_j} A_i \subseteq \bigcup_{0\le j'\le j-1} D_{j'}. \tag{2.1}$$

Proof. Assume on the contrary that there exist $u_i \in D_j$ and $i' \in A_i$ such that $u_{i'} \notin \bigcup_{0\le j'\le j-1} D_{j'}$. From the definition of $D_{j'}$, we know that there exists a path from u_0 to $u_{i'}$ with the length at least j. If u_i doesn't lie in this path, then we get a path from u_0 to u_i with the length at least $j+1$, since the direction of the edge $\overrightarrow{u_{i'}u_i}$ is from $u_{i'}$ to u_i. On the other hand, if u_i lies in this path, then clearly we get a directed cycle from u_i to $u_{i'}$, next to u_i. This also leads to a contradiction with the property (i) of Lemma 2.1. $\square$

Lemma 2.4. *$D_j \ne \emptyset$ for each $1 \le j \le l$.*

Proof. Clearly $D_l \ne \emptyset$. Let u_{i_l} be a vertex in D_l. Then there exists a directed path in $\overrightarrow{K}$ from u_0 to u_{i_l} with the length l. Suppose that this path is

$$u_0 \to u_{i_1} \to u_{i_2} \to \cdots \to u_{i_{l-1}} \to u_{i_l}.$$

We claim that $i_j \in D_j$ for each $1 \le j \le l$. We use induction on j. Clearly our assertion holds when $j = l$. Assume that $j < l$ and $i_{j+1} \in D_{j+1}$. Clearly $l_{i_j} \ge j$ since $u_0 \to u_{i_1} \to \cdots \to u_{i_j}$ is a directed path with the length j. On the other hand, by Lemma 2.3, we have

$$u_{i_j} \in A_{i_{j+1}} \subseteq \bigcup_{0\le j'\le j} D_{j'}.$$

Hence $l_{i_j} \le j$. So $l_{i_j} = j$ and $i_j \in D_j$. We are done. $\square$

We shall use induction on m to prove Theorem 1.2. The case $m = 1$ is trivial. Now assume that $m \ge 2$ and our assertion holds for $m-1$. That is, there exists a graph $G^{(m-1)} = (V^{(m-1)}, E^{(m-1)})$ with $\omega(G^{(m-1)}) = m-1$ satisfying that

$$\max_{1\le i\le k} \omega(G^{(m-1)}[V_i]) = m-1$$

for arbitrary partition $V_1, \dots, V_k$ of $V^{(m-1)}$.

First, we shall create n graphs $H_0, H_1, \dots, H_{n-1}$. H_0 is a graph only having a vertex x_0. For each $i \in D_1$, H_i is one copy of $G^{(m-1)}$. Similarly, for $2 \le j \le l$ and every $i \in D_j$, assuming $H_{i'}$ have been created for all $i' \in \bigcup_{0 \le j \le j-1} D_{j'}$, let H_i be

$$h_i := \prod_{i' \in A_i} |V(H_{i'})|$$

disjoint copies of $G^{(m-1)}$, where $V(H_{i'})$ denotes the vertex set of $H_{i'}$.

Next, we shall add some edges between the vertices of H_i and the vertices of $H_{i'}$, for $0 \le j < j' \le l$, $i \in D_j$ and $i' \in D_{j'}$. For every $i \in D_1$, we join x_0 and H_i, i.e., join x_0 and all vertices of H_i. Below we shall inductively add the edges incident with the vertices of H_i for every $2 \le j \le l$ and $i \in D_j$. Suppose that $2 \le j \le d$, $i \in D_j$ and $A_i = \{i'_1, \dots, i'_s\}$ with $i'_1 < \dots < i'_s$. Assume that we have added the edges between the vertices of $H_{i'_1}$ and $H_{i'_2}$, for every $0 \le j'_1 < j'_2 \le j-1$ and $i'_1 \in D_{j'_1}$, $i'_2 \in D_{j'_2}$. Recall that H_i is formed by h_i disjoint copies of $G^{(m-1)}$. Let ψ_i be an arbitrary $1-1$ projection from $V(H_{i'_1}) \times \dots \times V(H_{i'_s})$ to $\{1, \dots, h_i\}$, where $V(H_{i'_1}) \times \dots \times V(H_{i'_s})$ denotes the Cartesian product of $H_{i'_1}, \dots, H_{i'_s}$. Then for each $(w_1, \dots, w_s) \in V(H_{i'_1}) \times \dots \times V(H_{i'_s})$, we join the vertices $w_1, \dots, w_s$ to the $\psi_i(w_1, \dots, w_s)$-th copy of $G^{(m-1)}$ in H_i. Taking the above processes from $j = 2$ to l, we obtain the desired graph $G^{(m)} = (V^{(m)}, E^{(m)})$.

The remainder task is to show that G_m certainly satisfies our requirements. Clearly $\omega(G^{(m)}) \ge m$ since $\omega(G^{(m-1)}) = m-1$ and x_0 is adjacent to all vertices of at least one copy of $G^{(m-1)}$. Let Ω be an arbitrary complete subgraph of $G^{(m)}$. We need to prove that Ω has at most m vertices. Let U_i be the set of all vertices of Ω lying in H_i. Notice that for distinct i and i', if there exist $w \in H_i$ and $w' \in H_{i'}$ such that $ww' \in E^{(m)}$, then either $i \in A_{i'}$ or $i' \in A_i$, i.e., u_i and $u_{i'}$ are adjacent in the graph K. Since K doesn't contain any triangle, we have $|\{i : U_i \ne \emptyset\}| \le 2$. There is noting to do if Ω is completely contained in one H_i, since $\omega(H_i) = \omega(G^{(m-1)}) = m-1$. Suppose that there exist distinct i, i' such that $U_i, U_{i'} \ne \emptyset$. Without loss of generality, assume that $i' \in A_i$. Observe that distinct vertices of $H_{i'}$ are joint to distinct copies of $G^{(m-1)}$ in H_i. So we must have $|U_{i'}| = 1$. Hence

$$|V(\Omega)| = |U_i| + |U_{i'}| \le \omega(G^{(m-1)}) + 1 = m.$$

Now assume that the vertex set $V^{(m)}$ is split into k disjoint subsets $V_1, \dots, V_k$. Without loss of generality, we may assume that $x_0 \in V_1$. Let $U_{i,g}^{(t)}$ be the set of the common vertices of V_t and the g-th copies of $G^{(m-1)}$ in H_i. By the induction hypothesis, we know that

$$\max_{1 \le t \le k} \omega(G^{(m)}[U_{i,g}^{(t)}]) = \omega(G^{(m-1)}) = m-1$$

for every $1 \le i \le n$ and $1 \le t \le h_i$. For every $i \in D_1$, let $g_i = 1$,

$$t_i = \min\{1 \le t \le k : \omega(G^{(m)}[U_{i,1}^{(t)}]) = m-1\}$$

and arbitrarily choose a vertex $w_i \in U_{i,1}^{(t_i)}$. Below we shall determine g_i, t_i, w_i inductively for $2 \le j \le l$ and $i \in D_j$. Assume that $j \ge 2$ and we have determined g_i, t_i, w_i for all

$$i \in \bigcup_{1 \le j' \le j-1} D_{j'}.$$

Then for $i \in D_j$, supposing $A_i = \{i'_1, \dots, i'_s\}$ with $i'_1 < \cdots < i'_s$, let $g_i = \psi_i(w_{i'_1}, \dots, w_{i'_s})$,

$$t_i = \min\{1 \le t \le k : \omega(G^{(m)}[U_{i,g_i}^{(t)}]) = m - 1\}$$

and let w_i be an arbitrary vertex in $U_{i,g_i}^{(t_i)}$. In particular, we set $t_0 = 1$ and $w_0 = x_0$.

Now we shall color the vertices of K with k colors. For $0 \le i \le n-1$, let the vertex u_i be colored with the t_i-th color. Since K is not k-colorable, there exist distinct $0 \le i, i' \le n-1$ such that $t_i = t_{i'}$ and $u_i u_{i'} \in E_K$, i.e., either $i \in A_{i'}$ or $i' \in A_i$. Without loss of generality, assume that $i' \in A_i$. Notice that $w_{i'} \in U_{i',g_{i'}}^{(t_i)}$ and $w_{i'}$ is adjacent to all vertices of the g_i-th copies of H_i. Also, we have $G^{(m)}[U_{i,g_i}^{(t_i)}]$ contains an $(m-1)$-complete subgraph. Thus we get an m-complete subgraph of $G^{(m)}[U_{i,g_i}^{(t_i)} \cup \{w_{i'}\}]$, which is also a subgraph of $G^{(m)}[V_{t_i}]$. We are done. $\square$

3. Proof of Theorem 1.3

For a system $\mathcal{A} = \{a_t(n_t)\}_{t=1}^s$ and a graph $G = (V, E)$ with $V = \{v_1, \dots, v_s\}$, we say G is an intersection graph of $\mathcal{A}$ if

$$a_i(n_i) \cap a_j(n_j) \ne \emptyset \iff \text{the edge } v_i v_j \in E$$

for any $1 \le i < j \le s$. The following result [21, Theorem 1] is due to Zhang, although we give a slightly different proof here for the sake of completeness.

Lemma 3.1. *For each graph $G = (V, E)$ with $|V| = s$, there exists a system $\mathcal{A} = \{a_t(n_t)\}_{t=1}^s$ such that G is an intersection graph of $\mathcal{A}$.*

Proof. We use induction on s. The cases $s = 1$ and $s = 2$ are trivial. Assume that $s > 2$ and our assertion holds for $s - 1$. Suppose that $V = \{v_1, \dots, v_s\}$. Let $V' = V \setminus \{v_s\}$ and $G' = G[V']$. Let $\mathcal{A}' = \{a'_t(n'_t)\}_{t=1}^{s-1}$ be a system such that G' is an intersection graph of $\mathcal{A}'$. Let $p_1, \dots, p_{s-1}$ be some distinct primes greater than $\max\{n'_1, \dots, n'_{s-1}\}$. For each $1 \le t \le s-1$, let $n_t = n'_t p_t$ and a_t be an integer such that $a_t \equiv a'_t \pmod{n'_t}$ and $a_t \equiv 1 \pmod{p_t}$. Let $n_s = p_1 \cdots p_{s-1}$ and a_s be an integer such that

$$a_s \equiv \begin{cases} 1 \pmod{p_t} & \text{if the edge } v_t v_s \in E, \\ 0 \pmod{p_t} & \text{if the edge } v_t v_s \notin E \end{cases}$$

for $1 \le t \le s-1$. Since $a_i(n_i) \cap a_j(n_j) \ne \emptyset$ if and only if $(n_i, n_j) \mid a_i - a_j$, it is easy to see that G is an intersection graph of the system $\mathcal{A} = \{a_t(n_t)\}_{t=1}^s$. $\square$

Suppose that $G = (V, E)$ is an intersection graph of $\mathcal{A} = \{a_t(n_t)\}_{t=1}^s$. By the Chinese remainder theorem, for a subset $I \subseteq \{1, \dots, k\}$, if $a_i(n_i) \cap a_j(n_j) \neq \emptyset$ for any $i, j \in I$, then $\bigcap_{i \in I} a_i(n_i) \neq \emptyset$. Hence we have

$$\omega(G) = \max\{w_{\mathcal{A}}(x) : x \in \mathbb{Z}\},$$

by recalling that $w_{\mathcal{A}}(x) = |\{1 \leq i \leq s : x \in a_s(n_s)\}|$.

Proof of Theorem 1.3. Let $G = (V, E)$ be the graph satisfying the properties in Theorem 1.2 for $k = 2$. Assume that $|V| = s$. By Lemma 3.1, there exists a system $\mathcal{A} = \{a_t(n_t)\}_{t=1}^s$ such that G is an intersection graph of $\mathcal{A}$. We claim that for any partition $\{\mathcal{A}_1, \mathcal{A}_2\}$ of $\mathcal{A}$,

$$\max_{i=1,2} \omega_{\mathcal{A}_i} = \omega_{\mathcal{A}},$$

where

$$\omega_{\mathcal{A}} = \max\{w_{\mathcal{A}}(x) : x \in \mathbb{Z}\}.$$

In fact, letting $V_i \subseteq V$ be the set of vertices concerning those arithmetic progressions in $\mathcal{A}_i$, we have $G[V_i]$ is an intersection graph of $\mathcal{A}_i$. Hence

$$\max_{i=1,2} \omega_{\mathcal{A}_i} = \max_{i=1,2} \omega(G[V_i]) = \omega(G) = \omega_{\mathcal{A}}.$$

Since $\omega(G) = m$, $w_{\mathcal{A}}(x) \leq m$ for every $x \in \mathbb{Z}$. So we may choose integers $b_1, \dots, b_r$ such that $\mathcal{B} = \mathcal{A} \cup \{b_j(N)\}_{j=1}^r$ forms an exact m-cover, where N is the least common multiple of $n_1, \dots, n_s$. If $\mathcal{B}$ is arbitrarily split into $\mathcal{B}_1$ and $\mathcal{B}_2$, then

$$\max_{i=1,2} \omega_{\mathcal{B}_i} \geq \max_{i=1,2} \omega_{\mathcal{B}_i \cap \mathcal{A}} = \omega_{\mathcal{A}} = \omega_{\mathcal{B}}.$$

Hence there exists an integer x such that $w_{\mathcal{B}_1}(x) = m$ or $w_{\mathcal{B}_2}(x) = m$. Without loss of generality, assume that $w_{\mathcal{B}_1}(x) = m$. Then $w_{\mathcal{B}_2}(x) = w_{\mathcal{B}}(x) - w_{\mathcal{B}_1}(x) = 0$, whence $\mathcal{B}_2$ is not a 1-cover. □

4. A Further Remark

We may consider a general problem. Let $\mathscr{H}$ be a set of graphs such that for any $G \in \mathscr{H}$, all induced subgraphs of G are also contained in $\mathscr{H}$. Suppose that ψ be a projection from $\mathscr{H}$ to $\mathbb{N} = \{0, 1, 2, \dots\}$. We may ask whether for every $m \geq 0$ and $k \geq 2$, there exists a graph $G = (V, E) \in \mathscr{H}$ with $\psi(G) = m$ satisfying that

$$\psi(G) \in \{\psi(G[V_1]), \psi(G[V_2]), \psi(G[V_k])\}$$

for any k-partition $\{V_1, V_2, \dots, V_k\}$ of the vertex set V.

Let $l(G)$ denote the length of the longest path of G. Then we have the following negative result for $l(\cdot)$.

Theorem 4.1. *Let $G = (V, E)$ be a graph having at least one edge. Then there exists a partition $\{V_1, V_2\}$ of the vertex set V such that*

$$l(G[V_1]) < l(G)$$

and V_2 is an independent set.

Proof. Suppose that $l = l(G)$ and

$$\begin{aligned} L_1 &= x_{1,1} - x_{1,2} - \cdots - x_{1,l} \\ L_2 &= x_{2,1} - x_{2,2} - \cdots - x_{2,l} \\ &\cdots\cdots \\ L_t &= x_{t,1} - x_{t,2} - \cdots - x_{t,l} \end{aligned}$$

are all paths of G with the length l. Below we shall construct some sets U_i and I_i. Let $U_1 = \{x_{1,1}\}$ and

$$I_1 = \{1 \le i \le t : U_1 \cap L_i = \emptyset\}.$$

For $j \ge 2$, if $I_{j-1} \ne \emptyset$, then let $i' = \min I_{j-1}$, $U_j = U_{j-1} \cup \{x_{i',1}\}$ and

$$I_j = \{1 \le i \le t : U_j \cap L_i = \emptyset\}.$$

Of course, if $I_{j-1} = \emptyset$, then stop this process. Suppose that we finally get the vertex set U_s. Assume that $U_s = \{x_{i_1,1}, x_{i_2,1}, \ldots, x_{i_s,1}\}$ where $1 = i_1 < i_2 < \cdots < i_s$. Let $V_2 = U_s$ and $V_1 = V \setminus V_2$. First, we claim that V_2 is an independent set. Assume on the contrary that there exist $1 \le a < b \le s$ such that $x_{i_a,1}$ and $x_{i_b,1}$ are adjacent in G. By the construction of U_s, we have $x_{i_a,1}$ doesn't lie in the path L_{i_b}. Thus

$$x_{i_a,1} - x_{i_b,1} - x_{i_b,2} - \cdots - x_{i_b,l}$$

forms a path with the length $l+1$. It is impossible since $l(G) = l$. Second, by noting that $I_s = \emptyset$, we have $V_2 \cap L_i \ne \emptyset$ for any $1 \le i \le t$. Hence $l(G[V_1]) < l$ since $L_1, \ldots, L_t$ are all paths of G with the length l. $\square$

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